Discontinuous Galerkin methods for elliptic and hyperbolic problems

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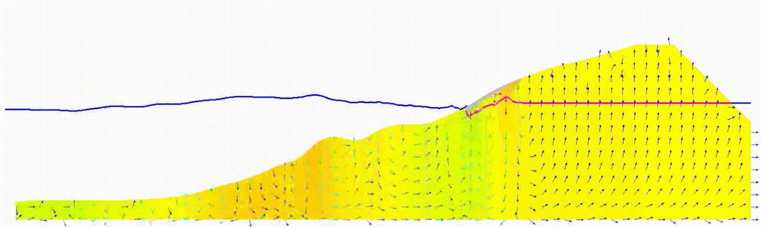
- Model behavior of water over and in a porous medium

⇒ Better understanding erosion and flooding phenomenon



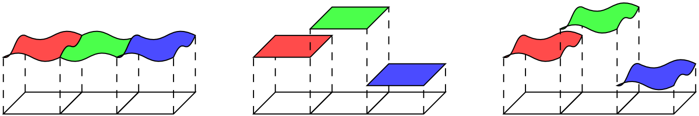
- Clément in 2021 developed RIVAGE, a Discontinuous Galerkin solver for Richards' equation
- Addressed Flow of water in the porous medium, one way coupling

⇒ Theoretical study of convergence for the DG solver for Richards' equation

⇒ Implement in RIVAGE a DG solver for a free surface model

⇒ Coupling with Richards' equation and Shallow water equations established by an asymptotic study

- Based on a variational formulation as in Finite Element Methods (FEM)
- Designed in an element-wise way as in Finite Volume Methods (FVM)



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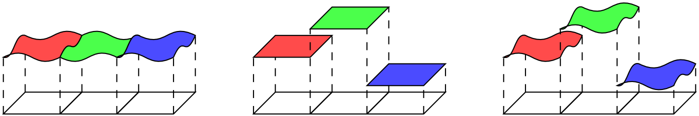

Elliptic problem :
Richards' Equation

- Close to FEM methods
- Use of user defined penalization parameters

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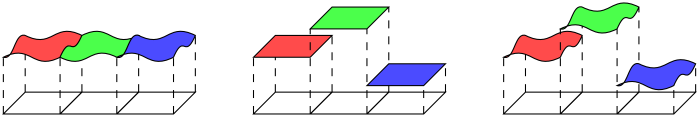

Elliptic problem : Richards' Equation

- Close to FEM methods
- Use of user defined penalization parameters

Hyperbolic problem : Shallow Water Equations

- Close to FVM methods
- Spurious oscillations
- Treatment of void problems

- They are derived from mass conservation and Darcy's law for a two-phase flow

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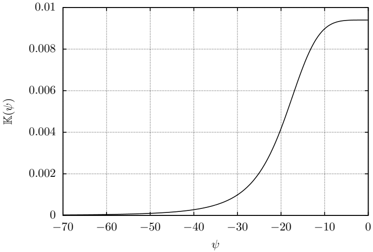
- Parabolic non-linear equation which describes flow in a porous medium

Richards' equation

$$\partial_t \theta(h - z) - \nabla \cdot (\mathbb{K}(h - z)\nabla h) = 0$$

- h : hydraulic head $[L]$
- z : elevation $[L]$
- $\psi = h - z$: pressure head $[L]$
- θ : water content $[\backsim]$
- $\mathbb{K}$: hydraulic conductivity $[L \cdot T^{-1}]$



Generic non-linear problem: steady state of Richards' equation

Let us consider the problem $(\mathcal{P})$ on the interval $\Omega = [a, b] \subset \mathbb{R}$:
For a given f in $L^2(\Omega)$, find $u(x) : \Omega \longrightarrow \mathbb{R}$ such that

$$(\mathcal{P}) \left\{ \begin{array}{rcll} -(K(x,u)u')' & = & f, & \text{in } \Omega \\ u & = & 0, & \text{on } \partial\Omega \end{array} \right.$$

$(\mathcal{P})$ can be cast into the weak formulation $(\mathcal{V})$

$$(\mathcal{V}): \quad \text{Find } u \in H_0^1(\Omega) \text{ such that,} \quad a(u,v;u) = l(v),\ \forall v \in H_0^1(\Omega)$$

with $a(u,v;u) = -\int_\Omega (K(x,u)u')'v\,dx$ and $l(v) = \int_\Omega f v\,dx$

Assuming that

$$(\mathcal{H}): \quad 0 < K_0 \leq K(x,u) \leq K_1, \quad \forall x \in \Omega,\ \forall u \in L^2(\Omega)$$

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- Non-linear weak formulation

$\Rightarrow$ Fixed point method to solve the non linear problem

$\Rightarrow$ Lax-Milgram theorem applied to the linearized problem

$\Rightarrow$ Discretization using Discontinuous Galerkin methods

It yields a linearized problem of $(\mathcal{V})$:

Operator T : For a given $\bar{u} \in L^2(\Omega)$,

$(\tilde{\mathcal{V}})$: Find $u \in H_0^1(\Omega)$ such that, $\quad \tilde{a}(u, v; \bar{u}) = l(v), \ \forall v \in H_0^1(\Omega)$

with $\tilde{a}(u, v; \bar{u}) = -\int_\Omega (K(x, \bar{u})u')'v dx$ and $l(v) = \int_\Omega f v dx$

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- $T(u) = u$ leads to the fixed-point method
- Proof of existence using Schauder fixed-point theorem
- Proof of uniqueness following the work of Boccardo, Gallouët and Murat[1]

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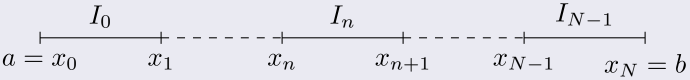
- Let $a = x_0 < ... < x_N = b$ be a mesh $\mathcal{E}_h$ of $\Omega = [a, b]$ and denote $I_n = (x_n, x_{n+1})$ a cell :



We define:

$$|I_n| = h = \frac{b-a}{N}, \quad \forall n \in \{0, .., N-1\}.$$

Let define the finite element subspace:

$$V_h^p = \left\{ v \in H_0^1(\Omega) \mid \forall I_n \in \mathcal{E}_h,\ v_{|I_n} \in \mathbb{P}_p(I_n) \right\}$$

the set of piecewise polynomials functions

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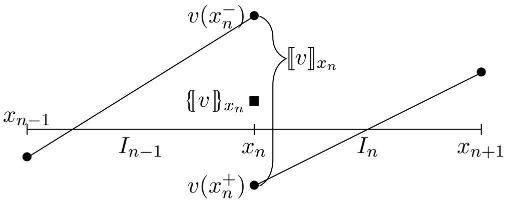
Define the jump and the average at x_n:

$$[\![v]\!]_{x_n} = v(x_n^-) - v(x_n^+), \quad \{\!\{v\}\!\}_{x_n} = \frac{1}{2}\left(v(x_n^-) + v(x_n^+)\right)$$



Take $(\tilde{\mathcal{V}})$ and take $u_h \in V_h^p$ and $v_h \in V_h^p$:

$$\tilde{a}(u_h, v_h) = l(v_h) \Leftrightarrow -\sum_{n=0}^{N-1} \int_{I_n} (K(x, \bar{u})u_h')' v_h dx = \int_\Omega f v_h dx$$

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Integrate by parts: Discontinuous Galerkin formulation $\forall v_h \in V_h^p$

$$\Leftrightarrow \sum_{n=0}^{N-1} \int_{I_n} K(x,\bar{u}) u_h' v_h' dx - \sum_{n=0}^{N-1} \left[K(x,\bar{u}) u_h' v_h \right]_{x_n^+}^{x_{n+1}^-} = \int_\Omega f v_h dx$$

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Integrate by parts: Finite Volume formulation $\forall v_h \in V_h^0$

$$\Leftrightarrow \cancel{\sum_{n=0}^{N-1} \int_{I_n} K(x,\bar{u})u_h' v_h' dx} - \sum_{n=0}^{N-1} \Big[K(x,\bar{u})u_h' v_h \Big]_{x_n^+}^{x_{n+1}^-} = \int_\Omega f v_h dx$$

- Incomplete Interior Penalty Galerkin (IIPG) method introduced by Dawson, Sun and Wheeler[2] in 2004.

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Rearrange the Discontinuous Galerkin formulation, assuming that $[\![K(x,\bar{u})u_h']\!]_{x_n} = 0$ and with penalization parameters σ_n:

$$\tilde{a}_h(u_h, v_h) = \sum_{n=0}^{N-1} \int_{I_n} K(x,\bar{u})u_h' v_h' dx - \sum_{n=0}^{N-1} \Big[K(x,\bar{u})u_h' v_h \Big]_{x_n^+}^{x_{n+1}^-}$$

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with $[\![ab]\!] = [\![a]\!]\{\!\{b\}\!\} + \{\!\{a\}\!\}[\![b]\!]$

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$$l_h(v_h) = \int_\Omega f v_h dx$$

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The discrete linearized problem $(\tilde{\mathcal{V}}_h)$ can now be defined:

$$(\tilde{\mathcal{V}}_h)\left\{ \quad \text{Find } u_h \in V_h^p \text{ such that}, \quad \tilde{a}_h(u_h, v_h) = l_h(v_h),\ \forall v_h \in V_h^p \right.$$

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Assuming that

$$(\mathcal{H}_h) \begin{cases} \exists K_0^{(n)}, K_1^{(n)} \in \mathbb{R}_+^*, \ \forall x \in I_n, \ K_0^{(n)} \leq K(x, \bar{u}) \leq K_1^{(n)} \\ \text{and } K_0 := \min_n K_0^{(n)} \text{ and } K_1 := \max_n K_1^{(n)} \end{cases}$$

Lemma (Existence and uniqueness of the discrete solution for the linearized discrete problem $(\tilde{\mathcal{V}}_h)$)

Consider $\bar{u} \in V_h^p$, *then* $\exists! u_h \in V_h^p$ *such that* $\tilde{a}_h(u_h, v_h) = l_h(v_h), \; \forall v_h \in V_h^p$

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We associate V_h^p with the norm:

$$\|v\|^2 = \sum_{n=0}^{N-1} \|v'\|_{I_n}^2 + \sum_{n=0}^{N} \frac{1}{h} [\![v]\!]_{x_n}^2 = \sum_{n=0}^{N-1} \|v'\|_{I_n}^2 + |v|_J^2$$

Where $\|\cdot\|_{I_n}$ is the usual norm $L^2(I_n)$ and $|v|_J^2 := \sum_{n=0}^{N} \frac{1}{h} [\![v]\!]_{x_n}^2$ is the jump semi-norm.

Following the work of Epshteyn and Rivière[3] we are able to prove

Lemma (Discrete coercivity of $\tilde{a}_h$)

For any vector of positive numbers $\epsilon = (\varepsilon^{(n)})_n$ and $\alpha > 0$, there exists a constant $C^*(\alpha, \epsilon) > 0$ such that $\forall u_h \in V_h^p,\ \tilde{a}_h(u_h, u_h) \geq C^*(\alpha, \epsilon)\|u_h\|^2$

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Lemma (Discrete continuity of l_h)

There exists a constant $B > 0$ *such that* $\forall v_h \in V_h^p,\ |l_h(v_h)| \leq B\|v_h\|$.

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Proofs give us :

- lower bounds for penalization parameters

$$\begin{cases} \forall n,\ \varepsilon^{(n)} < 2,\ \sigma_n = \alpha\sigma_n^* \\ \text{with } \alpha > 1 \end{cases} \quad \text{with} \quad \begin{cases} \forall n \in \{1, ..., N-1\}, \\ \sigma_n^* = \dfrac{(K_1^{(n)} C_{tr})^2}{2\varepsilon^{(n)} K_0^{(n)}}; \\ \sigma_0^* = \dfrac{(K_1^{(0)} C_{tr})^2}{\varepsilon^{(0)} K_0^{(0)}}; \\ \sigma_N^* = \dfrac{(K_1^{(N-1)} C_{tr})^2}{\varepsilon^{(N-1)} K_0^{(N-1)}}. \end{cases}$$

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- Expressions for $C^*(\alpha, \epsilon)$ and $\tilde{C}(\alpha, \epsilon)$

Following the work of Di Pietro and Ern published in 2011[4]

Theorem (Convergence to minimal regularity solutions)

Let $p \geq 1$, u_h be a sequence of approximate solutions generated by solving the discrete linearized problem $(\tilde{\mathcal{V}}_h)$ with penalty parameters ensuring coercivity. Then as $h \to 0$

$$
\begin{aligned}
& u_h \longrightarrow u \text{ strongly in } L^2(\Omega) \\
& u_h' \longrightarrow u' \text{ strongly in } L^2(\Omega) \\
& |u_h|_J \to 0
\end{aligned}
$$

where $u \in H_0^1(\Omega)$ is the unique solution of the problem $(\tilde{\mathcal{V}})$.

[4] *Di Pietro and Ern. 2011-11-03.*

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- Céa's lemma links C^* and $\tilde{C}$ to the approximation error

Lemma (Céa's lemma)

Let $u \in H_0^1(\Omega)$ *be the solution of* $(\tilde{\mathcal{V}})$ *and* u_h *the solution of* $(\tilde{\mathcal{V}}_h)$ *then* $\forall v \in H_0^1(\Omega)$ *we have :*

$$\|u - u_h\| \leq \gamma \|u - v\|,$$

with $\gamma(\alpha, \epsilon) = \dfrac{\tilde{C}(\alpha, \epsilon)}{C^*(\alpha, \epsilon)}$

Lemma (Céa's lemma)

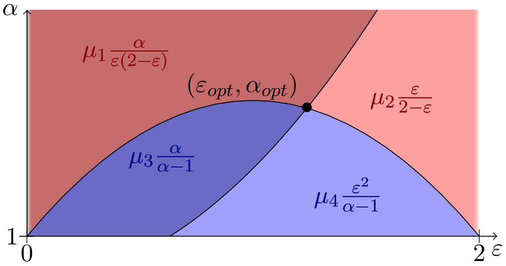
Let $u \in H_0^1(\Omega)$ *be the solution of* $(\tilde{\mathcal{V}})$ *and* u_h *the solution of* $(\tilde{\mathcal{V}}_h)$ *then* $\forall v \in H_0^1(\Omega)$ *we have :*

$$\|u - u_h\| \leq \gamma \|u - v\|,$$

with $\gamma(\alpha, \epsilon) = \dfrac{\tilde{C}(\alpha, \epsilon)}{C^*(\alpha, \epsilon)}$

- Find values for α and ε such that $\tilde{a}_h$ is coercive, continue and $\gamma(\alpha, \varepsilon)$ is minimal

For instance :
In the case of $\varepsilon^{(n)} = \varepsilon,\ \forall n$ and a certain configuration we seek min of these functions:



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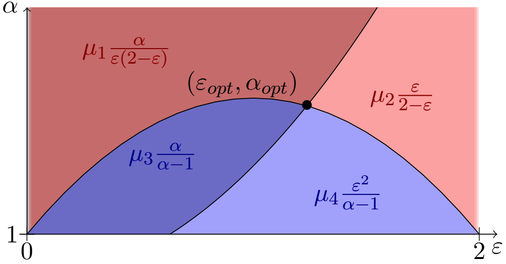

- We find $(\alpha_{opt}, \varepsilon_{opt}) \in]1, +\infty[\times]0, 2[$ such that γ is minimal

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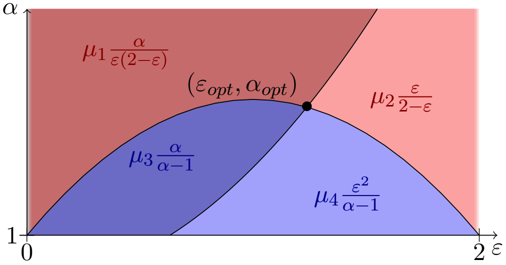

- We find $(\alpha_{opt}, \varepsilon_{opt}) \in]1, +\infty[\times]0, 2[$ such that γ is minimal
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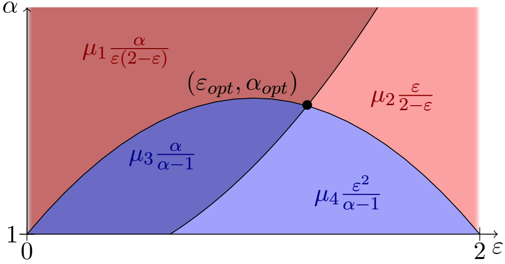

- We find $(\alpha_{opt}, \varepsilon_{opt}) \in]1, +\infty[\times]0, 2[$ such that γ is minimal
- α_{opt} and ε_{opt} are function of K_0 and K_1
- We can now find automatically penalization parameters with

$$\sigma_n = \alpha_{opt}\sigma_n^*(\varepsilon_{opt})$$

Haverkamp's test case

- Problem based on physical experiment[5]
- Infiltration in soil
- Modeled by Richards' equation using Vachaud's[6] relations

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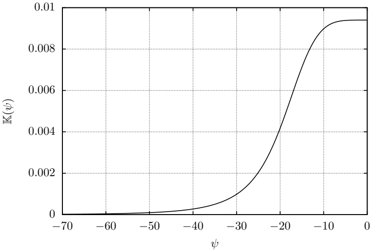
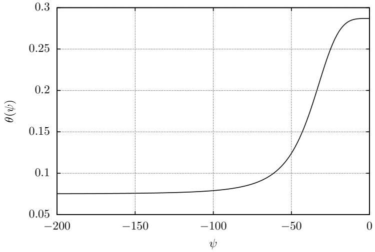


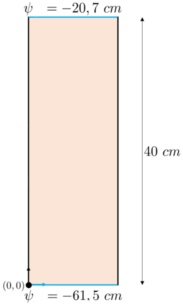


Haverkamp's test case

- Problem based on physical experiment[5]
- Infiltration in soil
- Modeled by Richards' equation using Vachaud's[6] relations

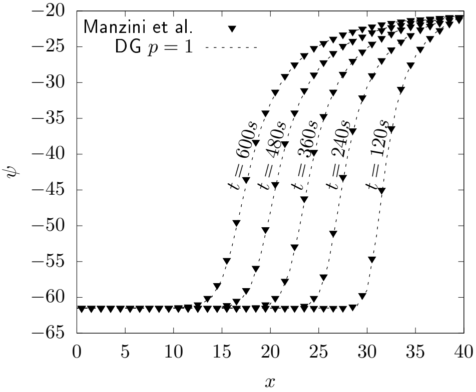
Find $\psi(x,t) : [0,40] \times [0,T] \longrightarrow \mathbb{R}$ such that

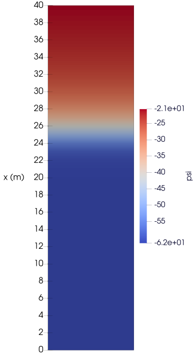
$$\begin{cases} \partial_t\theta(\psi) - \partial_x(\mathbb{K}(\psi))\partial_x(\psi + x)) = 0 & , \text{ in }]0,40[\times[0,T] \\ \psi(z,0) = -61.5 & , \text{ in }]0,40[\\ \psi(0,t) = -61.5 & , \text{ in } [0,T] \\ \psi(40,t) = -20.7 & , \text{ in } [0,T] \end{cases}$$



- Piecewise linear approximation, $\Delta x = 1$
- Time integration with backward Euler method

Haverkamp's test case





- Good agreement with Manzini et al.[7] VF methods

[7] *Manzini and Ferraris. Advances in Water Resources. 2004-12.*

Haverkamp's test case

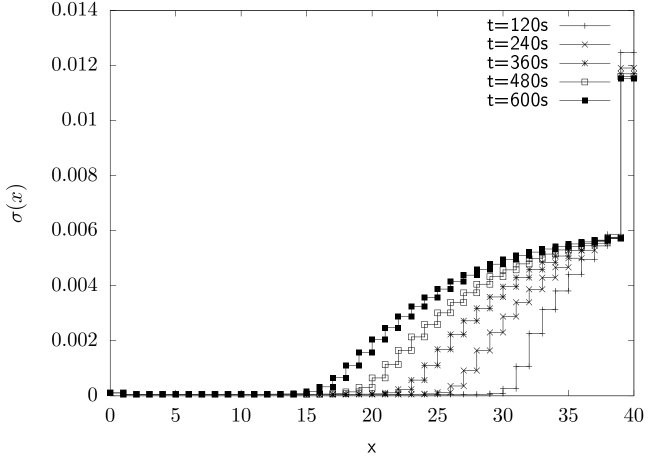


Figure: Penalization parameters plot for the numerical solution

- Addressed the problem of penalization parameters values
  - Auto calibrated
  - Not increase condition number
  - Minimize error

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- Proved that the whole loop of resolution converges to the unique weak solution

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  - Auto calibrated
  - Not increase condition number
  - Minimize error

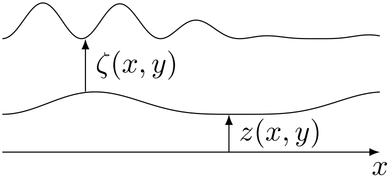
- Proved that the whole loop of resolution converges to the unique weak solution

- Developed a one dimensional code and validated it

$\Rightarrow$ Implement auto calibration of penalization parameters in 2D and 3D

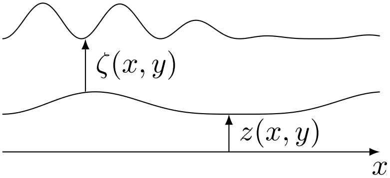
$$\begin{cases} \partial_t \begin{pmatrix} \zeta \\ q_x \\ q_y \end{pmatrix} + \nabla \cdot \begin{pmatrix} q_x & q_y \\ \dfrac{q_x^2}{\zeta} + g\dfrac{\zeta^2}{2} & \dfrac{q_x q_y}{\zeta} \\ \dfrac{q_x q_y}{\zeta} & \dfrac{q_y^2}{\zeta} + g\dfrac{\zeta^2}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ -g\zeta\partial_x z \\ -g\zeta\partial_y z \end{pmatrix} \quad \text{in } \Omega\times]0,T[, \\ \text{Initial and Boundary conditions,} \end{cases}$$

- Depth-averaged incompressible Navier-Stokes Equations
- Hyperbolic system

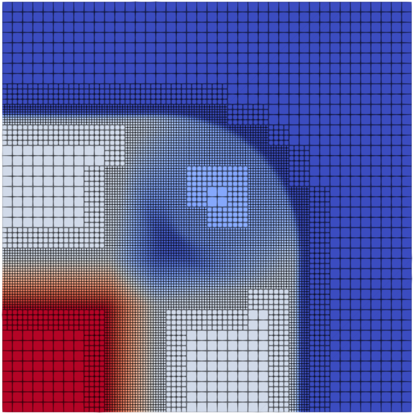


$$\begin{cases} \partial_t U + \nabla \cdot \mathbb{G}(U) = \mathbb{S}(U, z) \text{ in } \Omega\times]0, T[, \\ \text{Initial and Boundary conditions,} \end{cases}$$

- Depth-averaged incompressible Navier-Stokes Equations
- Hyperbolic system



Space discretization: the mesh $\mathcal{E}_h$



- Unstructured mesh
- Non conformal mesh
- Mesh adaptation along calculation

Adaptation criterion:

- $\nabla\zeta$
- Production of numerical entropy

Space discretization: variational formulation

Solution space: $V_h^p = \left\{ v \in H_0^1(\Omega) \mid \forall I_n \in \mathcal{E}_h,\ v_{|I_n} \in \mathbb{P}_p(I_n) \right\}$ the set of piecewise polynomials functions

- $p = 0$ Finite volume methods: piecewise constant
- $p = 1$ Piecewise linear and so on

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Find $U_h := (\zeta_h, (q_x)_h, (q_y)_h) \in [V_h^p(E)]^3$ such that $\forall t \in]0, T[$,

$$\begin{cases} \partial_t U_h + \nabla \cdot \mathbb{G}(U_h) = \mathbb{S}(U_h, z_h), \\ \text{Initial and Boundary conditions,} \end{cases}$$

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$\forall t \in]0, T[$, $\forall \varphi_h \in [V_h^p(E)]^3$ and

$$\begin{cases} \varphi_h \partial_t U_h + \varphi_h \nabla \cdot \mathbb{G}(U_h) = \varphi_h \mathbb{S}(U_h, z_h), \\ \text{Initial and Boundary conditions,} \end{cases}$$

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$$\begin{cases} \displaystyle\int_E \varphi_h \partial_t U_h + \int_E \varphi_h \nabla \cdot \mathbb{G}(U_h) = \int_E \varphi_h \mathbb{S}(U_h, z_h), \\ \text{Initial and Boundary conditions,} \end{cases}$$

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$$\begin{cases} \displaystyle\int_E \varphi_h \partial_t U_h - \int_E \nabla\varphi_h : \mathbb{G}(U_h)^T + \sum_{F \in \mathcal{F}_h^E} \int_F \varphi_h \hat{G}_F(U_h) = \int_E \varphi_h \mathbb{S}(U_h, z_h) \\ \text{Initial condition} \end{cases}$$

Time discretization

$U_h|_E$ and φ_h linear combination of polynomial: $\forall (x,y), t \in E\times]0,T]$

$$U_h|_E(x,y,t) = \mathbf{\Phi}(x,y)\cdot \mathbf{U}_E(t) \text{ and } \varphi_h(x,y) = \mathbf{\Phi}(x,y)$$

$$\underbrace{\int_E \mathbf{\Phi}\otimes\mathbf{\Phi}}_{\mathbb{M}_E} \frac{d\mathbf{U}_E}{dt} = \underbrace{\int_E \nabla\mathbf{\Phi} : \mathbb{G}(U_h)^t - \sum_{F\in\mathcal{F}_h^E} \int_F \mathbf{\Phi}\hat{G}_F(U_h) + \int_E \mathbf{\Phi}\mathbb{S}(U_h, z_h)}_{\mathcal{H}_E(U_h(t))}$$

Time discretization

$$\mathbb{M}_E \frac{d\mathbf{U}_E}{dt} = \mathcal{H}_E(U_h(t))$$

Explicit Runge-Kutta method of order $q = p + 1$:

- Δt chosen according to CFL condition linked to[8]

$$\max_{E \in \mathcal{E}_h} \left(\frac{\lambda_E}{h_E}\right) \Delta t \leq \frac{1}{2p+1}$$

Legendre basis makes mass matrix diagonal and ease analytical calculus

[8] *Cockburn and Shu. Mathematics of Computation. 1989.*

Well balanced property

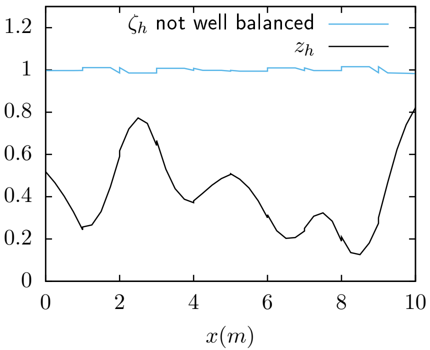
Solving Shallow Water equation with the previous RKDG method does not preserve equilibrium states:

- $\zeta + z \equiv C$ a constant and $\mathbf{q} \equiv 0$

- ζ_h and z_h in V_h^p admit jumps at elements' interfaces

- Bathymetry is defined as solution, on each elements

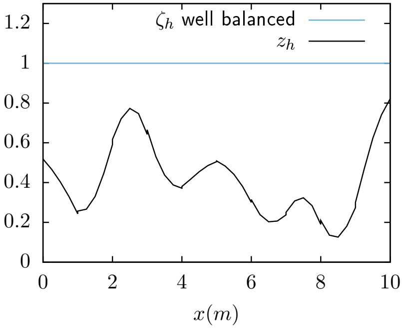
- Numerical fluxes no longer equal to zero



Hydrostatic reconstruction

Variational formulation modified[9] such that interfaces flux cancels out if $\zeta + z \equiv C$

$$\int_E \varphi_h \frac{\partial U_h}{\partial_t} - \int_E \nabla \varphi_h : \mathbb{G}(U_h)^t$$
$$+ \sum_{F \in \mathcal{F}_h^E} \int_F \varphi_h \left(\hat{G}_F(U_h^\diamond) - \delta_F(U_h, z_h) \right)$$
$$= \int_E \varphi_h \mathbb{S}(U_h, z_h)$$

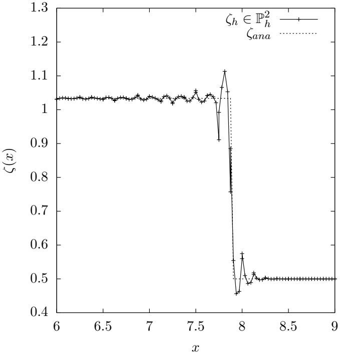


[9] *Ern, Piperno, and Djadel. International Journal for Numerical Methods in Fluids. 2007.*

Moment limiting

Spurious oscillations around discontinuities, due to:

- Hyperbolic problem
- High order scheme ($p > 0$)



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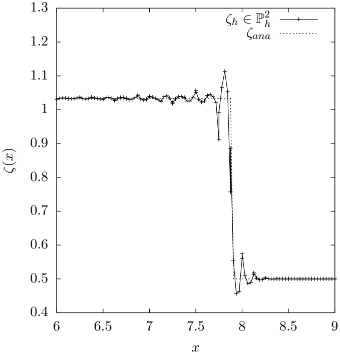

- Post processing
- Well suited for non conformal mesh

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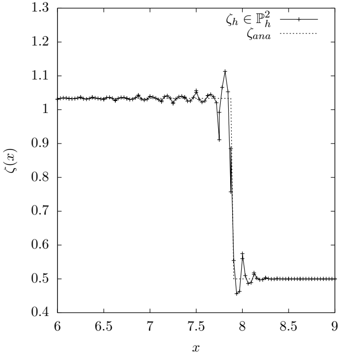


- Post processing
- Well suited for non conformal mesh

For each element[a]:

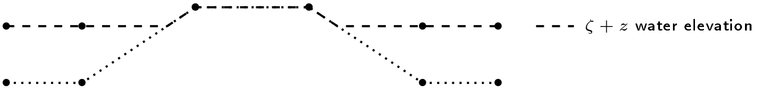
1. Estimate n-th derivative with $(n-1)$-th derivative of surrounding elements
2. Minmod comparison with the computed n-th derivative of the element

[a] *Krivodonova. Journal of Computational Physics. 2007-09.*

- Loss of hyperbolicity if $\zeta_h \leq 0$

$\Rightarrow$ Need to preserve positivity of water depth

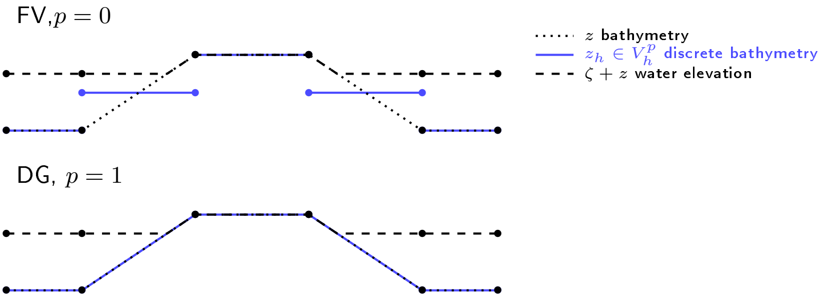
FV,$p = 0$



DG, $p = 1$

- Loss of hyperbolicity if $\zeta_h \leq 0$

$\Rightarrow$ Need to preserve positivity of water depth



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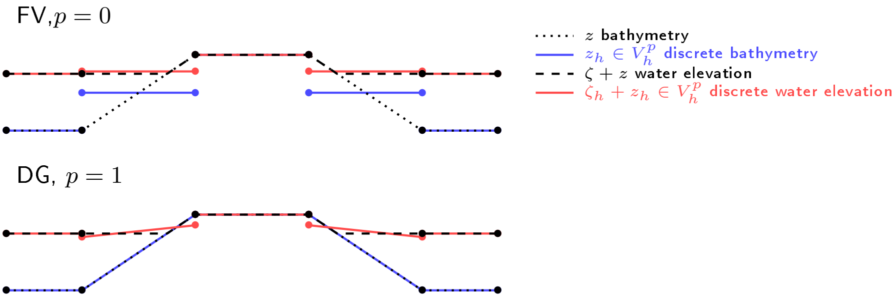
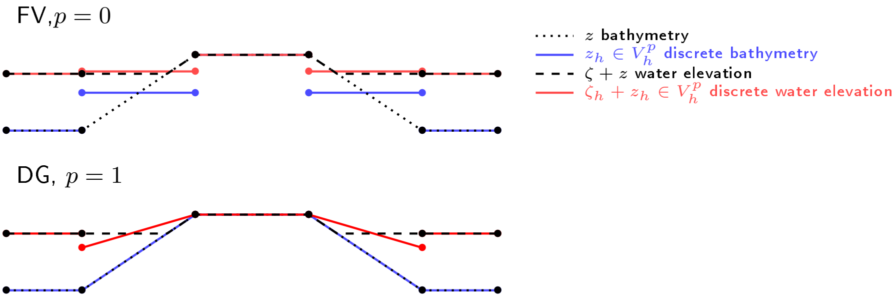


- Semi-dry cells in DG, $p = 1$

- Loss of hyperbolicity if $\zeta_h \leq 0$

$\Rightarrow$ Need to preserve positivity of water depth



- Semi-dry cells in DG, $p = 1$

$\Rightarrow$ Use of post processing to treat dry cells and semi-dry cells

1D Dam-Break

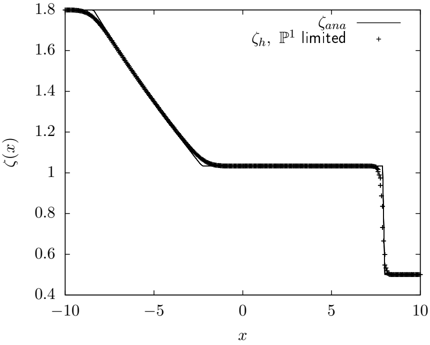


Figure: Water depth $\zeta_h \in \mathbb{P}_h^1$ compared with ζ_{ana}

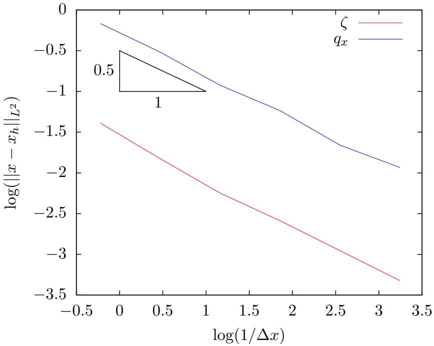
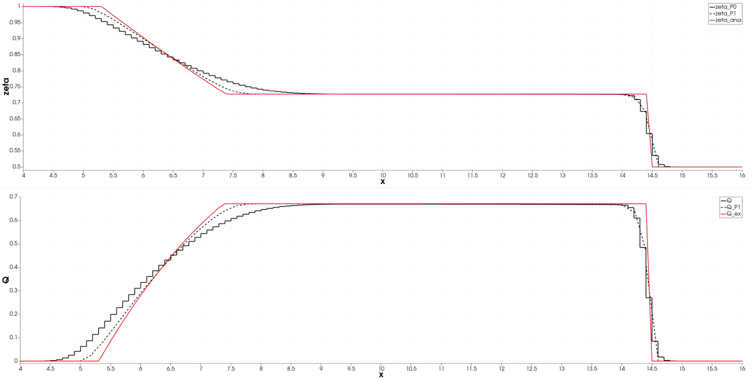


Figure: L^2-errors on $U_h \in \mathbb{P}_h^2$ with moment limiter

- Comparison between $p = 0$ and $p = 1$ limited for the same amount of degrees of freedom



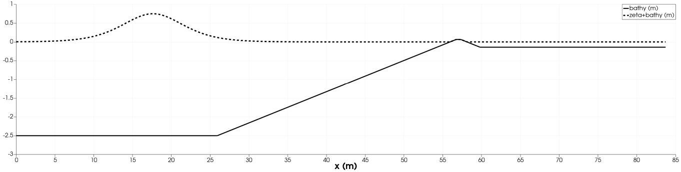
$\Rightarrow$ Same precision around choc area

$\Rightarrow$ Better contact discontinuity definition

Solitary wave propagation over a two-dimensional reef

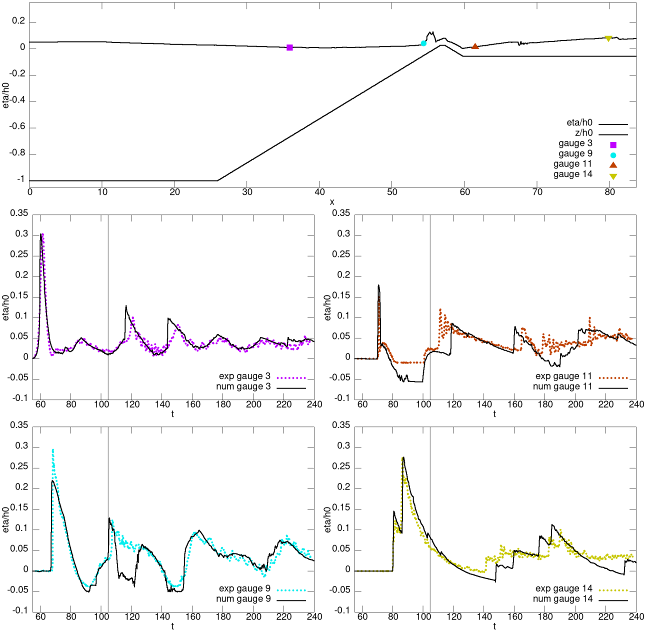
Experimental test case over a typical reef configuration[10]

- $83.7m$ long and $h_0 = 2.5m$ deep channel
- $1/12$ reef slope with a crest $0.065m$ above water level



- Piecewise cubic approximation ($p = 3$), Runge Kutta method of order 4
- 500 elements, $\Delta x = 0.1674$

[10] *Roeber, Cheung, and Kobayashi. Coastal Engineering. 2010.*


eta/h0
z/h0
gauge 3
gauge 9
gauge 11
gauge 14
x
t
exp gauge 3
num gauge 3
exp gauge 11
num gauge 11
exp gauge 9
num gauge 9
exp gauge 14
num gauge 14

- Solve Shallow Water Equations with RKDG methods
- Ensure well balanced property
- Cancel spurious oscillations on a non-conformal and unstructured mesh
- Solve flooding and drying problem

Elliptic

- Auto-calibration of penalization parameters
- Converges to the unique weak solution

Hyperbolic

- Ensure well balanced property
- Moment limiting and positive depth operator

- Implement auto-calibration of penalization parameters in higher dimensions
- Asymptotic model coupling Richards' equation and Shallow Water Equations