Runge-Kutta Discontinuous Galerkin method applied to Shallow Water Equations

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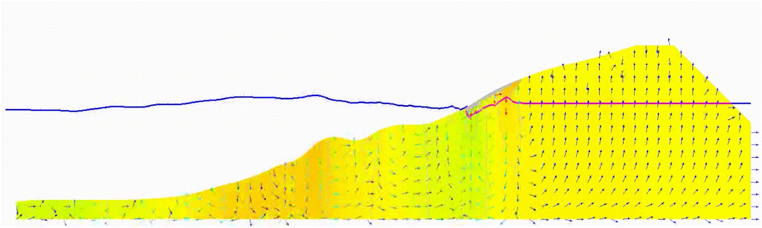
Topical Problems of Fluid Mechanics 2023

[1] *Université de Toulon, IMATH, EA 2137, 83957 La Garde, France*

- Model behavior of water over and in a porous medium

⇒ Better understanding erosion and flooding phenomenon



- Clément in 2021[1] developed RIVAGE, a Discontinuous Galerkin solver for Richards' equation

⇒ Addressed the flow of water in the porous medium

⇒ One way coupling

- Implement in RIVAGE a DG solver for a free surface model

⇒ Two way coupling

[1] *Clément et al. Advances in Water Resources. 2021.*

1. Governing equations
2. Runge-Kutta Discontinuous Galerkin methods
3. Numerical results
4. Conclusions and Perspectives

Shallow Water Equations

$$\begin{cases} \partial_t \begin{pmatrix} \zeta \\ q_x \\ q_y \end{pmatrix} + \nabla \cdot \begin{pmatrix} q_x & q_y \\ \dfrac{q_x^2}{\zeta} + g\dfrac{\zeta^2}{2} & \dfrac{q_x q_y}{\zeta} \\ \dfrac{q_x q_y}{\zeta} & \dfrac{q_y^2}{\zeta} + g\dfrac{\zeta^2}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ -g\zeta\partial_x z \\ -g\zeta\partial_y z \end{pmatrix} \quad \text{in } \Omega\times]0,T[, \\ \text{Initial and Boundary conditions,} \end{cases}$$

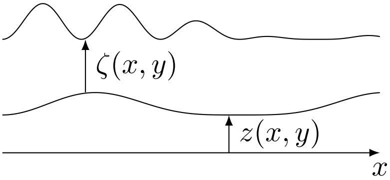
- Depth-averaged incompressible Navier-Stokes Equations[2]
- Hyperbolic system

[2] *Saint-Venant. 1871.*

Shallow Water Equations

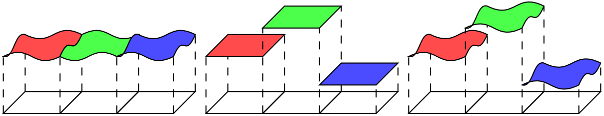
$$\begin{cases} \partial_t U + \nabla \cdot \mathbb{G}(U) = \mathbb{S}(U, z) \text{ in } \Omega \times]0, T[, \\ \text{Initial and Boundary conditions,} \end{cases}$$

- $U := (\zeta, \mathbf{q}) : \Omega \times [0, T[\to \mathbb{R}^3$ conservatives variables
- z bathymetry



Discontinuous Galerkin methods

- Based on a variational formulation as in Finite Element Methods (FEM)
- Designed in an element-wise way as in Finite Volume Methods (FVM)



Motivation

- Local method: mesh and order adaptation
- Designed to easily increase approximation order

Drawbacks with SWE

- Spurious oscillations
- Shore line definition
- High number of degrees of freedom

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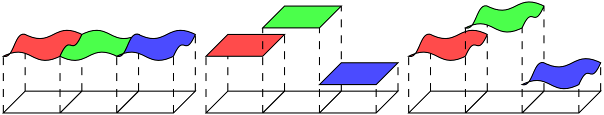

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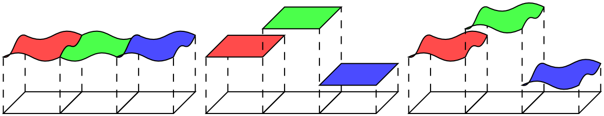

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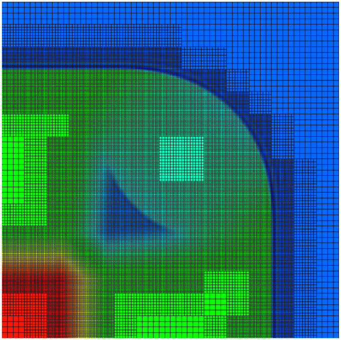
Space discretization: the mesh $\mathcal{E}_h$



- Unstructured mesh
- Non conformal mesh
- Mesh adaptation along calculation

Adaptation criterion:

- $\nabla\zeta$
- Production of numerical entropy

Space discretization: variational formulation

Solution space: $\mathbb{P}_h^p$ the set of piecewise polynomials functions

- $p = 0$ Finite volume methods: piecewise constant
- $p = 1$ Piecewise linear and so on

Find $U_h := (\zeta_h, (q_x)_h, (q_y)_h) \in [\mathbb{P}_h^p(E)]^3$ such that
$\forall t \in]0, T[$, $\forall E \in \mathcal{E}_h$ and $\forall \varphi_h \in [\mathbb{P}_h^p(E)]^3$

$$\begin{cases} \partial_t U_h + \nabla \cdot \mathbb{G}(U_h) = \mathbb{S}(U_h, z_h), \\ \text{Initial and Boundary conditions,} \end{cases}$$

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$$\begin{cases} \displaystyle\int_E \varphi_h \partial_t U_h + \int_E \varphi_h \nabla \cdot \mathbb{G}(U_h) = \int_E \varphi_h \mathbb{S}(U_h, z_h), \\ \text{Initial and Boundary conditions,} \end{cases}$$

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$\forall t \in]0, T[$, $\forall E \in \mathcal{E}_h$ and $\forall \varphi_h \in [\mathbb{P}_h^p(E)]^3$

$$\begin{cases} \displaystyle\int_E \varphi_h \partial_t U_h - \int_E \nabla\varphi_h : \mathbb{G}(U_h)^T + \sum_{F \in \mathcal{F}_h^E} \int_F \varphi_h \hat{G}_F(U_h) = \int_E \varphi_h \mathbb{S}(U_h, z_h) \\ \text{Initial condition} \end{cases}$$

Time discretization

$U_h|_E$ and φ_h linear combination of polynomial: $\forall (x, y), t \in E \times]0, T]$

$$U_h|_E(x, y, t) = \mathbf{\Phi}(x, y) \cdot \mathbf{U}_E(t) \text{ and } \varphi_h(x, y) = \mathbf{\Phi}(x, y)$$

$$\underbrace{\int_E \mathbf{\Phi} \otimes \mathbf{\Phi}}_{\mathbb{M}_E} \frac{d\mathbf{U}_E}{dt} = \underbrace{\int_E \nabla \mathbf{\Phi} : \mathbb{G}(U_h)^t - \sum_{F \in \mathcal{F}_h^E} \int_F \mathbf{\Phi} \hat{G}_F(U_h) + \int_E \mathbf{\Phi} \mathbb{S}(U_h, z_h)}_{\mathcal{H}_E(U_h(t))}$$

Time discretization

$$\mathbb{M}_E \frac{d\mathbf{U}_E}{dt} = \mathcal{H}_E(U_h(t))$$

Explicit Runge-Kutta method of order $q = p + 1$:

- Δt chosen according to CFL condition linked to
  - polynomial order p
  - eigenvalues of the problem
  - the mesh

Polynomial basis affect the shape of mass matrix

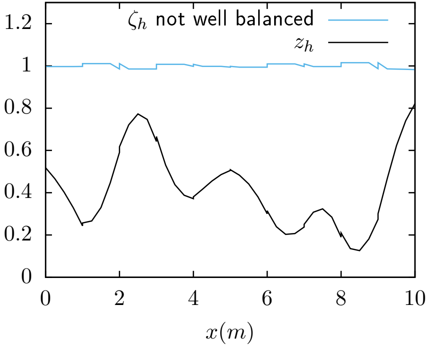
Well balanced property

Solving Shallow Water equation with the previous RKDG method does not preserve equilibrium states:

- $\zeta + z \equiv C$ a constant and $\mathbf{q} \equiv 0$

- ζ_h and z_h in $\mathbb{P}_h^p$ admit jumps at elements' interfaces

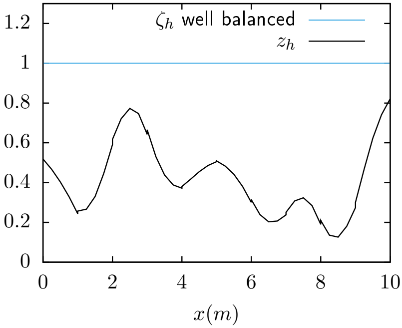
- Numerical fluxes no longer equal to zero



Hydrostatic reconstruction

Variational formulation modified[2] such that interfaces flux cancels out if $\zeta + z \equiv C$

$$\int_E \varphi_h \frac{\partial U_h}{\partial_t} - \int_E \nabla \varphi_h : \mathbb{G}(U_h)^t$$
$$+ \sum_{F \in \mathcal{F}_h^E} \int_F \varphi_h \left(\hat{G}_F(U_h^\diamond) - \delta_F(U_h, z_h) \right)$$
$$= \int_E \varphi_h \mathbb{S}(U_h, z_h)$$

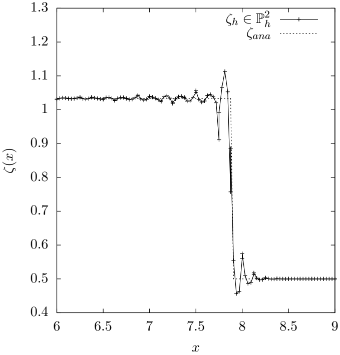


[2] *Ern, Piperno, and Djadel. International Journal for Numerical Methods in Fluids. 2007.*

Moment limiting

Spurious oscillations around discontinuities, due to:

- Hyperbolic problem
- High order scheme ($p > 0$)



- Post processing
- Well suited for non conformal mesh

For each element[a]:

1. Estimate n-th derivative with $(n-1)$-th derivative of surrounding elements
2. Minmod comparison with the computed n-th derivative of the element

[a] *Krivodonova. Journal of Computational Physics. 2007-09.*

1D Dam-Break

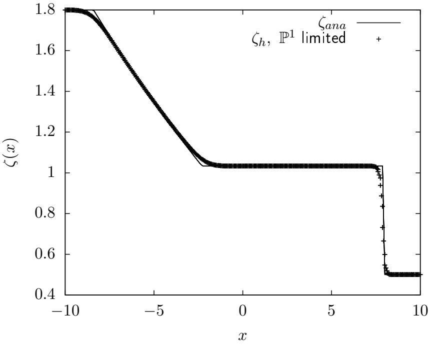


Figure: Water depth $\zeta_h \in \mathbb{P}_h^1$ compared with ζ_{ana}

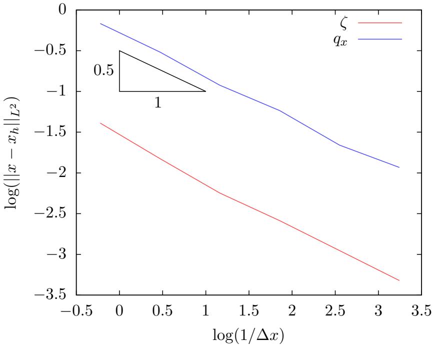


Figure: L^2-errors on $U_h \in \mathbb{P}_h^2$ with moment limiter

1D Dam-Break

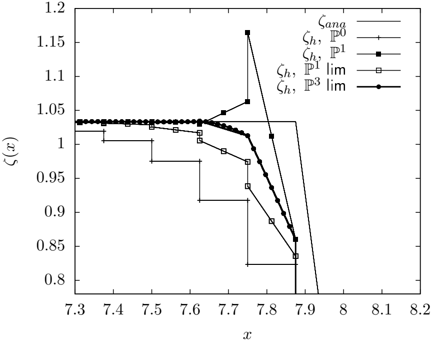
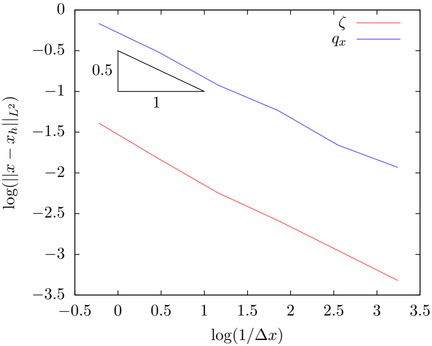


Figure: Zoom on solution discontinuity with different computation methods



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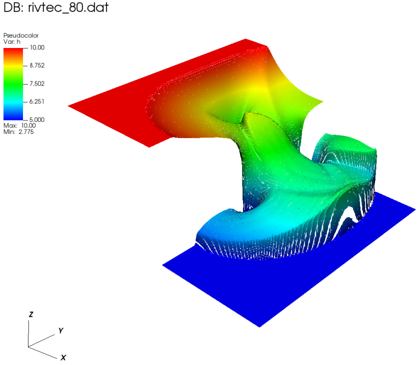
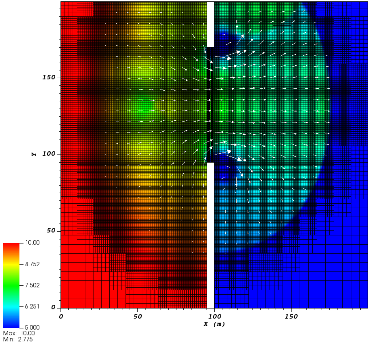
2D Dam-Break with breach

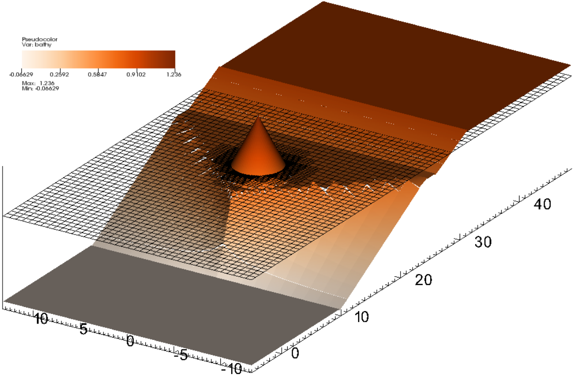




Approximation of the solution with $U_h \in \mathbb{P}_h^1$ and moment limiter[3]

[3] *Delis and Katsaounis. Applied Mathematical Modelling. 2005-08.*

Solitary wave propagation over a three-dimensional reef



Three-dimensional reef based on laboratory experiments[4]

[4] *Lynett et al. Coastal Engineering Proceedings. 2011.*

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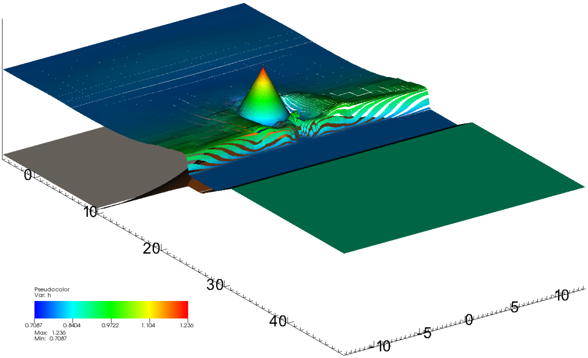

Approximation of the solution with $U_h \in \mathbb{P}_h^1$ and moment limiter[4]

[4] *Pons. Theses. 2018-12.*

- Solve Shallow Water Equations with RKDG methods
- Ensure well balanced property
- Cancel spurious oscillations on a non-conformal and unstructured mesh
- Solve flooding and drying problem[5]

- Enhance the flooding and drying method
- Implement moment limiting on triangular elements
- Strong coupling between Richards' equation and Shallow Water Equations

[5] *Lee and Lee. KSCE Journal of Civil Engineering. 2015.*