Discontinuous Galerkin method applied to Shallow Water Equations with flooding and drying treatment

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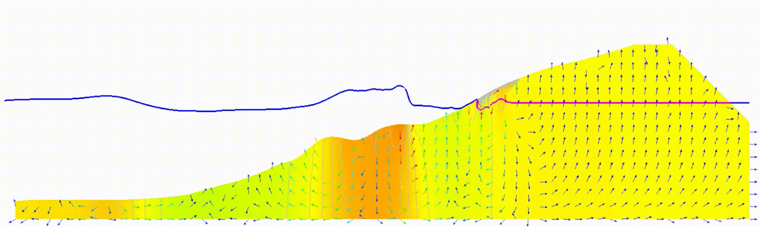
Topical Problems of Fluid Mechanics 2024

- Model behavior of water over and in a porous medium

$\Rightarrow$ Better understanding erosion and flooding phenomenon



- Clément in 2021[1] developed RIVAGE, a Discontinuous Galerkin (DG) solver for Richards' equation.
  - One way coupling
- We implemented in 2023[2], in RIVAGE, a DG solver for Shallow Water Equations

⇒ Implement in RIVAGE a DG solver for Shallow Water Equations with dry areas

⇒ Two way coupling

[1] *Clément et al. 2021 Advances in Water Resources.*

[2] *Poussel et al. 2023*

Shallow Water Equations (SWE) derived from :

- Incompressible Navier Stokes
- Hydrostatic approximation
- Depth average

$$\begin{cases} \partial_t h + \operatorname{div}(\mathbf{q}) = 0, \\ \partial_t \mathbf{q} + \operatorname{div}\left(\frac{\mathbf{q}\otimes\mathbf{q}}{h} + \frac{gh^2}{2}\mathbb{I}\right) = -gh\nabla z_b, \end{cases}$$

with proper boundary and initial conditions.

$\Rightarrow$ Hyperbolic system

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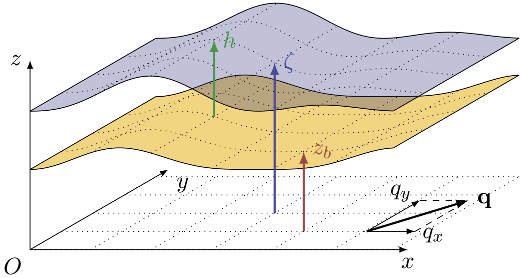
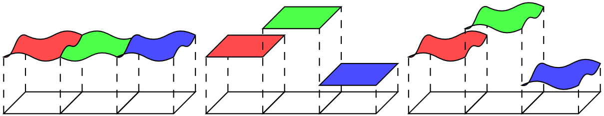


- h : water height $[m]$
- z_b : bathymetry elevation $[m]$
- $\zeta = h + z_b$: free surface elevation $[m]$
- $\mathbf{q} = (q_x, q_y)^T$: horizontal discharge $[m^2.s^{-1}]$

Discontinues Galerkin formulation

- Based on a variational formulation as in Finite Element Methods (FEM)
- Designed in an element-wise way as in Finite Volume Methods (FVM)



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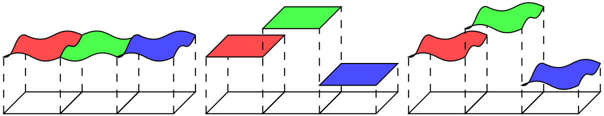

Motivation

- Local method: mesh and order adaptation
- Designed to easily increase approximation order

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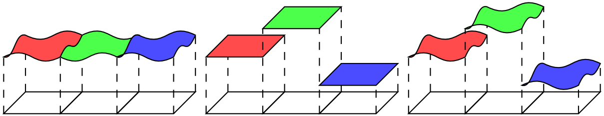

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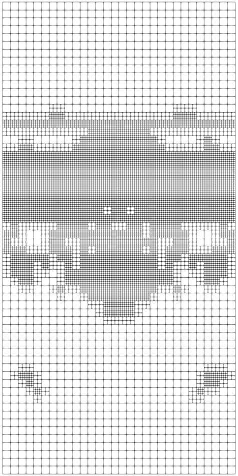
Drawbacks with SWE

- Spurious oscillations
- Moving shoreline definition
- High number of degrees of freedom

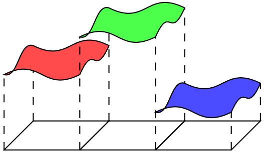
Discontinues Galerkin formulation



Discontinuous Galerkin (DG) space :

$$\mathcal{V}^p(\mathcal{E}) := \left\{ v : \Omega \to \mathbb{R} \;\middle|\; v_{|_E} \in \mathbb{P}^p(E),\ \forall E \in \mathcal{E} \right\}$$

- Set of piecewise polynomial functions



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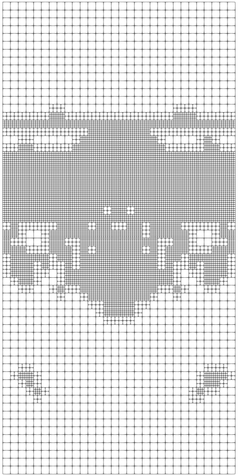

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- Set of piecewise polynomial functions

Find $\mathbf{U} := (h, \mathbf{q}) \in \mathcal{V}^p(\mathcal{E})$ such that $\forall t \in]0, T[$, $\forall \varphi \in \mathcal{V}^p(\mathcal{E})$ and $\forall E \in \mathcal{E}$

$$\int_E \varphi \partial_t \mathbf{U} - \int_E \partial_{x_i} \varphi \mathbf{G}_i(\mathbf{U}) + \sum_{F \in \mathcal{F}^E} \int_F \varphi \hat{G}_F(\mathbf{U}) = \int_E \varphi \mathbf{S}(\mathbf{U}, z_b)$$

Vectorial form of the DG SWE formulation:

$$\text{Find } \mathcal{U} : [0,T] \to \mathbb{R}^{N_{dof}} \text{ such that}$$

$$\forall t \in [0,T], \ \mathbb{M}\frac{d\mathcal{U}(t)}{dt} = \mathcal{H}(\mathcal{U}(t), z_b)$$

with $\mathbb{M}$ the mass matrix and $\mathcal{H}$ the DG operator. $\mathbb{M}$ is bloc diagonal.

- Explicit Runge-Kutta method of order $p+1$

$\Rightarrow$ CFL condition

Moment limiting and Well-balanced property

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- Explicit Runge-Kutta method of order $p + 1$

$\Rightarrow$ CFL condition

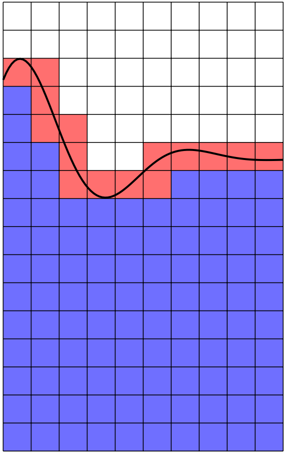
- Preserve equilibrium states, e.g. *lake at rest*
  - $\Rightarrow$ Modify the DG operator $\mathcal{H}$ to be well-balanced[a]
- Cancel spurious oscillations
  - $\Rightarrow$ *Moment limiting* post-processing[b]

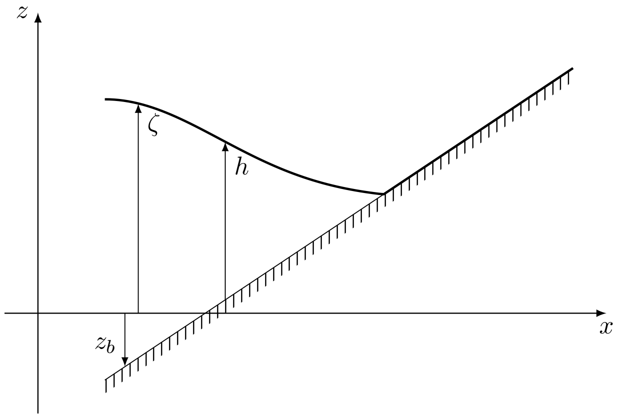
$$\mathbb{M}\frac{d\mathcal{U}(t)}{dt} = \Lambda\left(\mathcal{H}^{\mathsf{wb}}(\mathcal{U}(t), z_b)\right)$$

[a] *Ern, Piperno, and Djadel. 2007 International Journal for Numerical Methods in Fluids.*

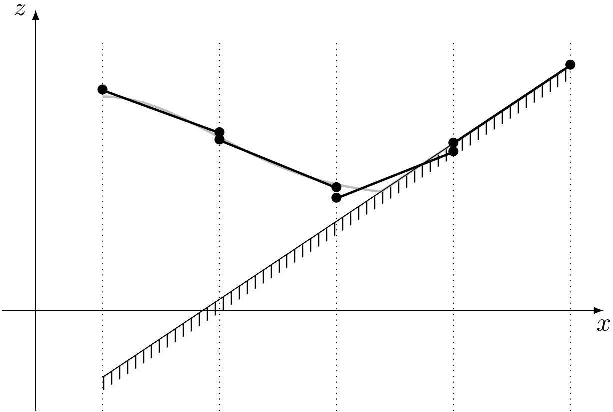
[b] *Krivodonova. 2007 Journal of Computational Physics.*

- Treat problem with dry area and moving shoreline
  - $\Rightarrow$ Dry (□), wet (■) and semi-dry elements (■)
- Loss of hyperbolicity when $h < 0$

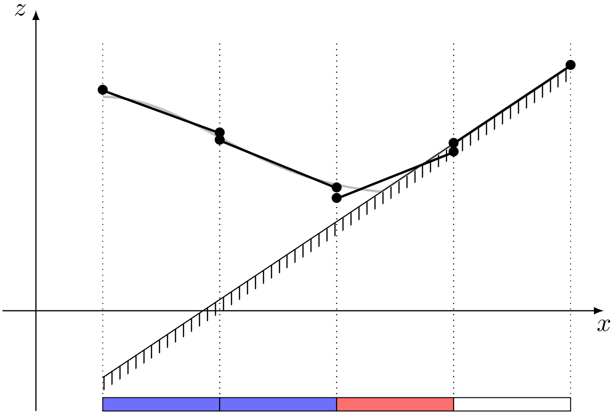




- Typical situation of shoreline



- Typical situation of shoreline
- Project ζ and z_b on the piecewise linear basis



- Typical situation of shoreline
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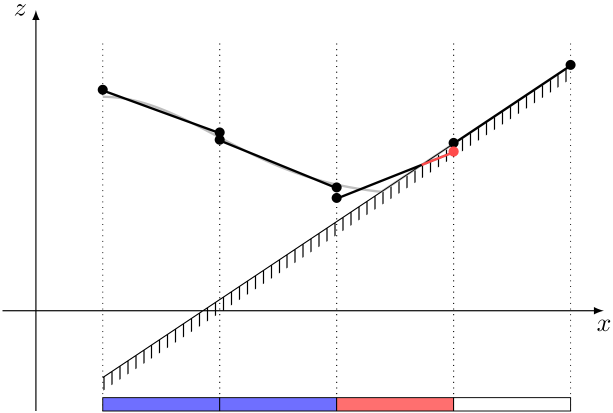
$\Rightarrow$ Elements are either wet (■), semi-dry (■) and dry (□)



- Typical situation of shoreline
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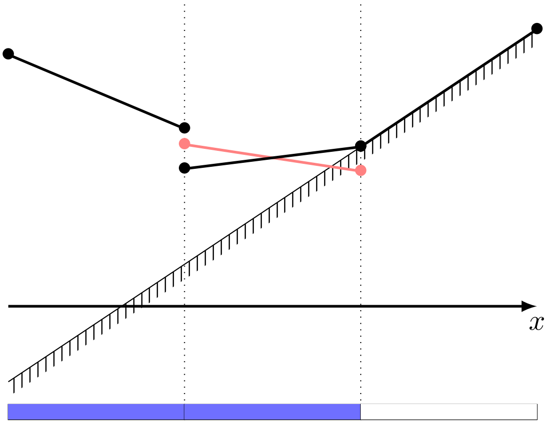
$\Rightarrow$ Elements are either wet (■), semi-dry (■) and dry (□)

$\Rightarrow$ Loss of hyperbolicity on the semi-dry element

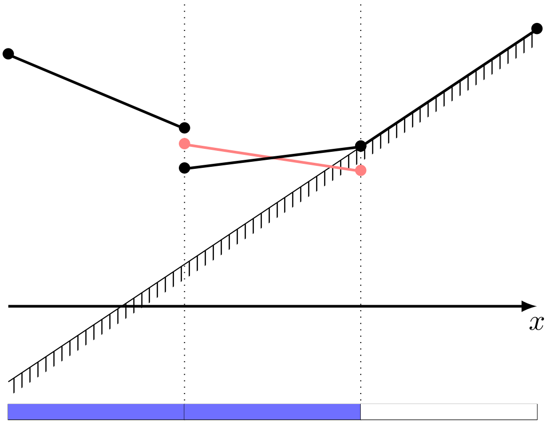
Slope modification



- Modify the slope of the water surface wihtout modifying the bathymetry[a]
- Modify discharge to be null on the shoreline

[a] *Xing, Zhang, and Shu. 2010 Advances in Water Resources.*

Slope modification



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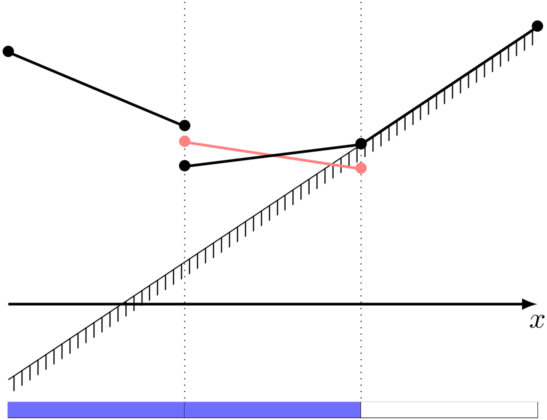

- Modify the slope of the water surface wihtout modifying the bathymetry[a]
- Modify discharge to be null on the shoreline

$\Rightarrow$ No loss of hyperbolicity at the interface

$\times$ No more well balanced on the shoreline

$\times$ Introduce non physical fluid speed

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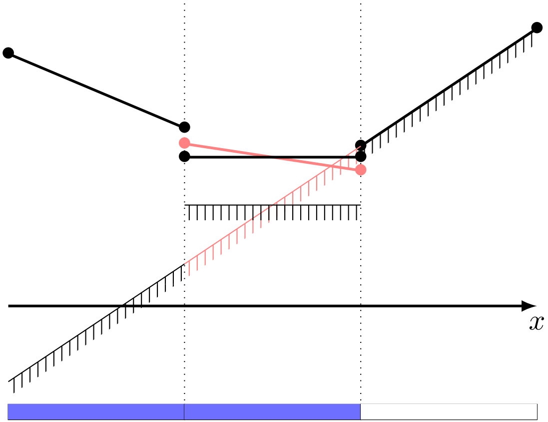
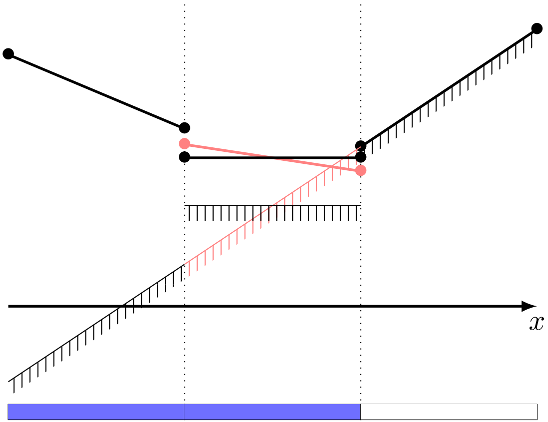
$\mathbb{P}^0$-adaptation



- Consider the solution piecewise constant on the shoreline

$\mathbb{P}^0$-adaptation



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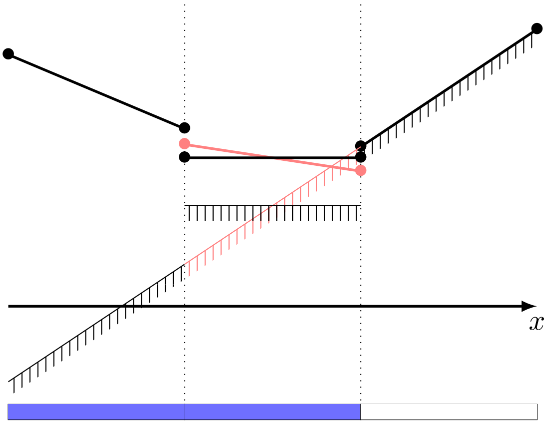


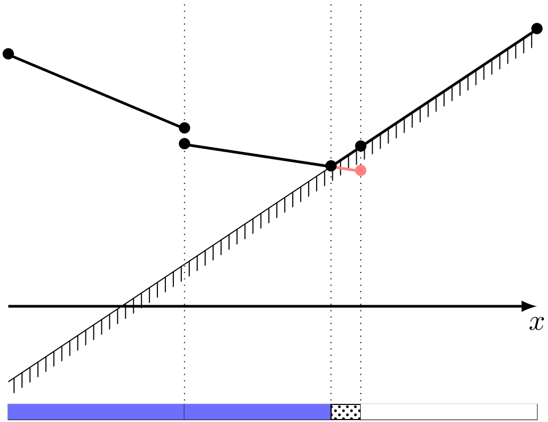
- Consider the solution piecewise constant on the shoreline

$\Rightarrow$ No loss of hyperbolicity at the interface

× No more well balanced on the shoreline

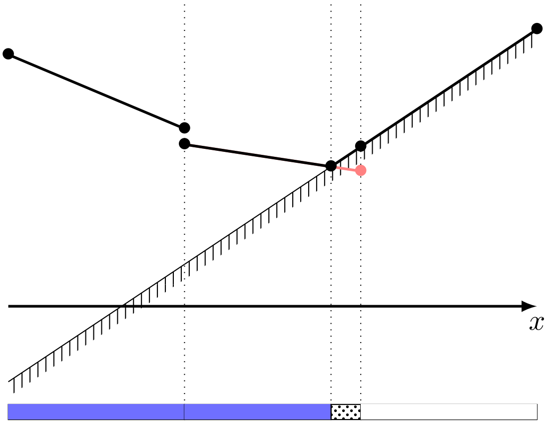
× Introduce non physical fluid speed

Ghost cells



- Virtually split the element on the shoreline, introduce a ghost cell (⚄)
  - The mesh is not modified !
  - The support of basis functions is modified

Ghost cells



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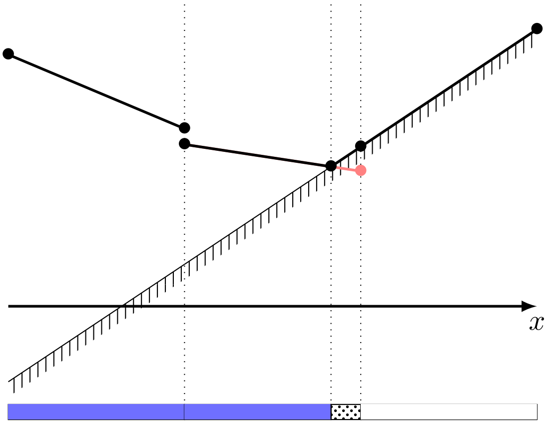

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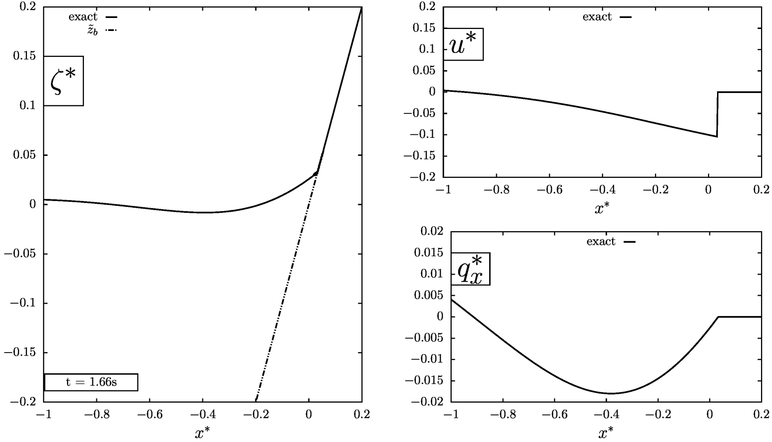
× Introduce non physical fluid speed

Uni-dimensional - Carrier & Greenspan test case

Monochromatic wave running up and down on a beach plane[3]



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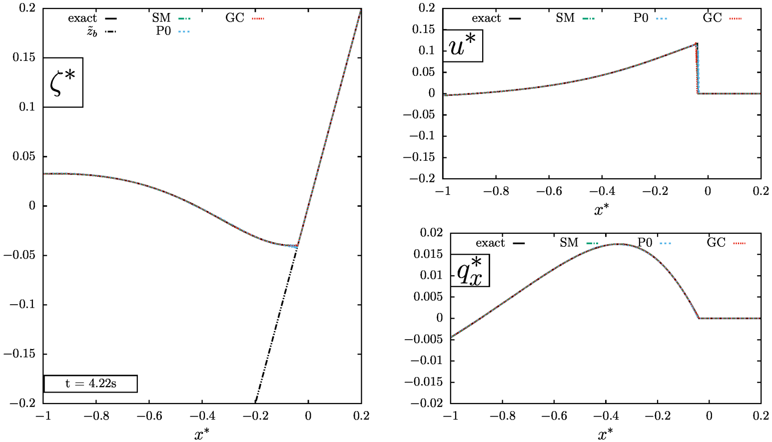
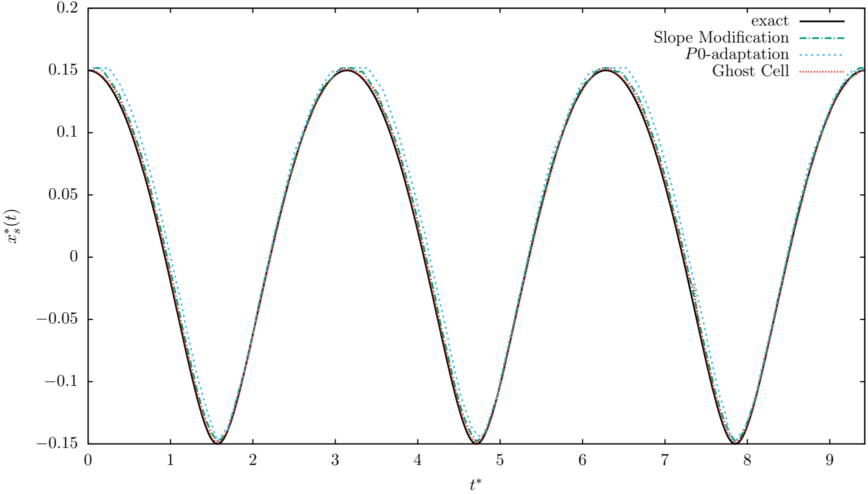
[3] *Carrier and Greenspan. 1958*
Journal of Fluid Mechanics.

Uni-dimensional - Carrier & Greenspan test case

Numerical solution with $p = 1$ and $\Delta x = 0.06m$



Comparison the recovered shoreline position



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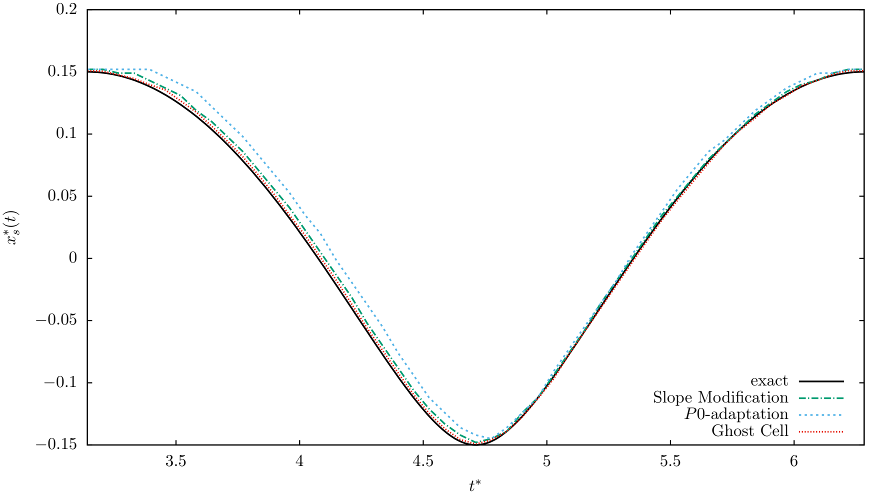
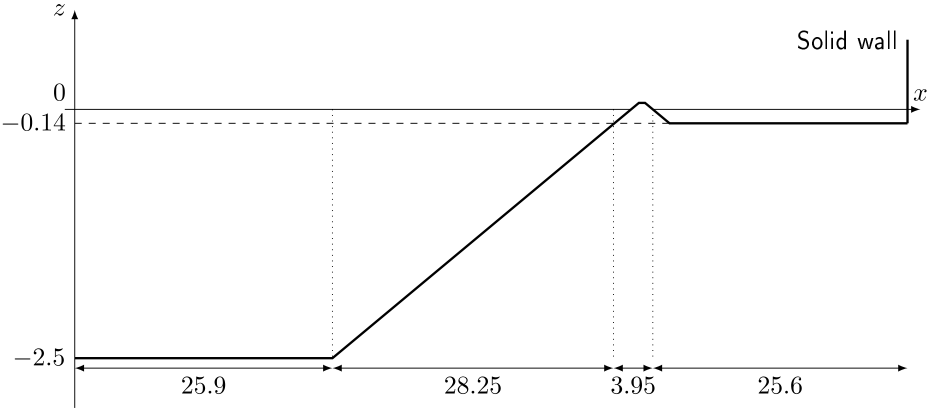


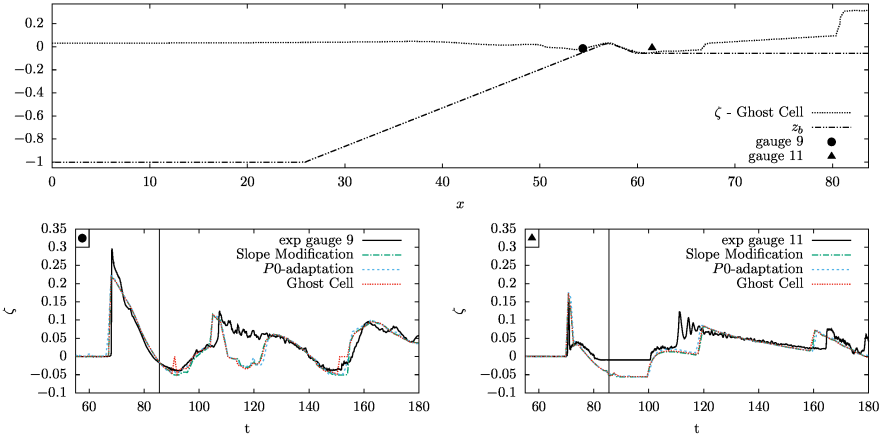
Uni-dimensional - Roeber test case

Solitary wave breaking over a two-dimensional reef[3]

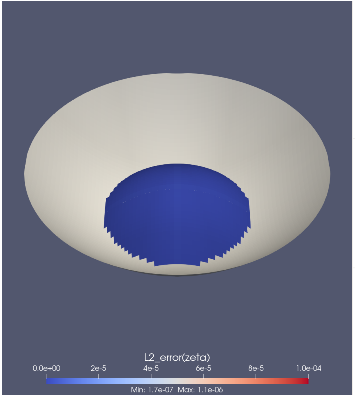


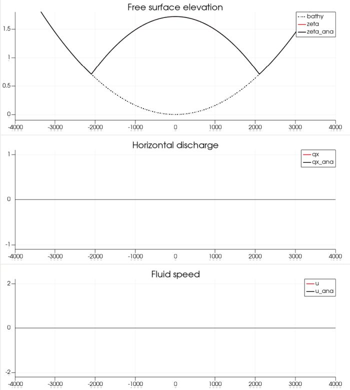
[3] *Roeber, Cheung, and Kobayashi. 2010 Coastal Engineering.*

Numerical solution with $p = 1$ and $\Delta x = 0.42m$



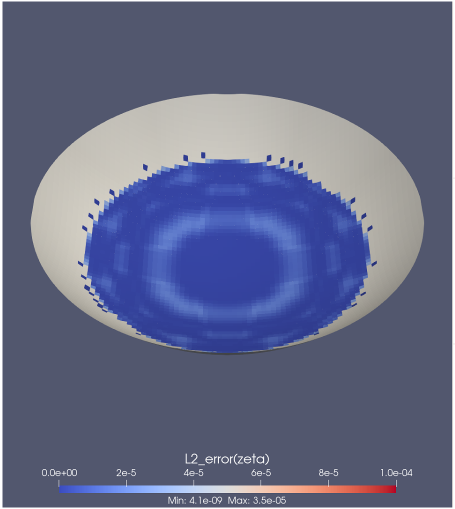
Two-dimensional axisymmetric phenomena[3] with $p = 1$, $\Delta x = 100m$ and *Slope Modification*

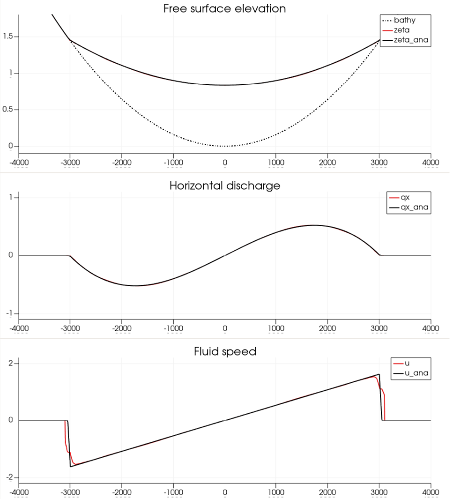




[3] *Bunya et al. 2009*
Computer Methods in Applied Mechanics and Engineering.

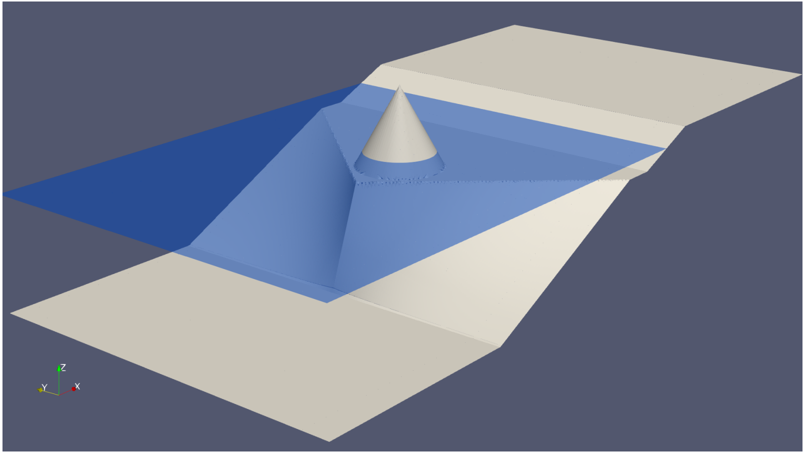
Bi-dimensional - Parabolic bowl test case





Bi-dimensional - Conical island test case

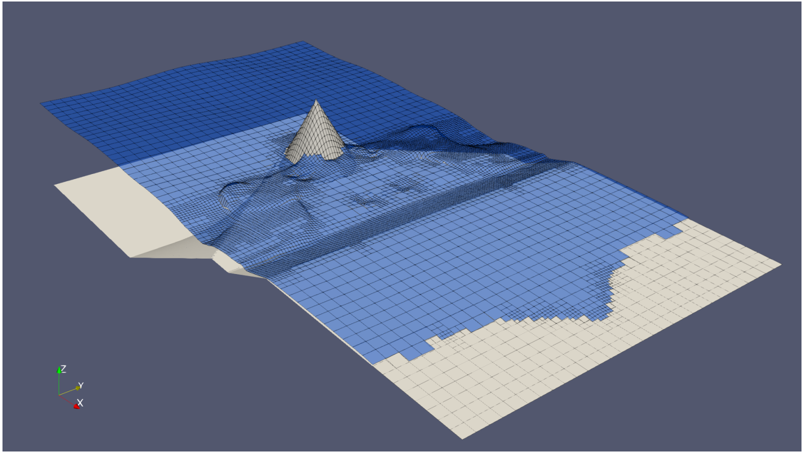
Solitary wave breaking over a three-dimensional reef[3]



[3] *Roeber and Cheung. 2012 Coastal Engineering.*

Bi-dimensional - Conical island test case

Numerical solution with $p = 1$, $\Delta x = 0.42m$ and *Slope Modification*



- Solve Shallow Water equations with **Discontinuous Galerkin discretization**
- Solve **problem with dry areas** in one and two dimensions

- Implement and validate in **three drying treatment**
- **Ghost Cell method performs better than usual methods**

- Make coexisting **adaptative mesh refinement** and drying treatment

$\Rightarrow$ Work on coupling Richards' equation and Shallow Water equations